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Three-spin giant magnons in $AdS_5 \times S^5$

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Abstract: From the Polyakov string action using a conformal gauge we construct a three-spin giant magnon solution describing a long open string in $AdS_5 \times S^5$ which rotates both in two angular directions of S^5 and in one angular direction of AdS_5. Through the Virasoro constraints the string motion in AdS_5 takes an effect from the string configuration in S^5. The dispersion relation of the soliton solution is obtained as a superposition of two bound states of magnons. We show that there is a correspondence between a special giant magnon in AdS_2 and the sinh-Gordon soliton.

Keywords: Bosonic Strings, Penrose limit and pp-wave background, AdS-CFT Correspondence.

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1. Introduction

According to the AdS/CFT correspondence [1] the string theory in $AdS_5 \times S^5$ should be dual to the $\mathcal{N} = 4$ super Yang-Mills (SYM) theory. The spectrum of certain string states matches with the spectrum of dimensions of field theory operators in the SYM theory [2, 3]. There has been a mounting evidence that the spectrum of AdS/CFT is described by studying the multi-spin rotating string solutions in $AdS_5 \times S^5$ [4, 5] and by analyzing the Bethe equation for the diagonalization of the integrable spin chain in the SYM theory [6–9]. The direct relation between both sides has been investigated at the level of effective action [10, 11] and the direct equivalence between the Bethe equation for the spin chain and the classical Bethe equation for the classical $AdS_5 \times S^5$ string sigma model has been shown from the view point of integrability [12, 13].

Recently, Hofman and Maldacena [14] have made a particular limit such that both the spin chain and the string effectively become very long and constructed a rotating open string solution in $R \times S^2$, namely, the giant magnon using the Nambu-Goto string action, which is a particular case of the spiky string in $R \times S^2$ [15] that is the generalization of the spiky string in AdS_3 [16], and is identified with an elementary magnon excitation in the long spin chain. The dispersion relation between energy and angular momentum J_1 for the giant magnon has been calculated to be equal to the strong 't Hooft coupling limit of the dispersion relation for the spin chain magnon that was derived by using the SU(2|2) $\times$ SU(2|2) supersymmetry with a novel central extension [17]. From the equivalence between the string theory in $R \times S^2$ and the sine-Gordon theory [18, 19], the giant magnon has been identified with the sine-Gordon soliton and the scattering phase of two magnons has been computed to be in agreement with the strong 't Hooft coupling limit of the conjecture of [20].

Analyzing the pole of the two-particle S-matrix [21] and exploiting the equivalence between the string theory in $R \times S^3$ and the complex sine-Gordon theory [22], the dispersion relation for the two-charge dyonic giant magnon has been presented to be described in terms of infinite J_1 and finite J_2 of angular momentum in an orthogonal plane. This two-spin giant magnon is interpreted as a bound state composed of J_2 magnons. From the Polyakov action the two-spin giant magnon solution has been constructed for the string theory on $R \times S^3$ in the conformal gauge and the finite-size effects for the string theory on $R \times S^2$ in the uniform gauge have been studied [23]. The dispersion relation for the two-spin giant magnon has been also derived from the Nambu-Goto action in the static gauge and the one-loop superstring correction to the known folded and circular two-spin string solutions has been analyzed in taking the limit of infinite J_1 with finite J_2 [24].

For the string theory in the β-deformed $AdS_5 \times S^5$ background the two-spin giant magnon solution has been presented [25, 26]. Using the relation with the complex sine-Gordon theory, a family of closed string solutions with two spins on $R \times S^3$ have been constructed [27] such that they interpolate the rotating two-spin closed strings [4] and the dyonic giant magnons.

Applying the dressing method [28] to the SO(6) vector model describing strings in $R \times S^5$ the three-spin giant magnon solution specified by infinite J_1 and finite J_2 and J_3 has been derived [29] as a state consisting of two superposed, noninteracting, two-charge bound states, one with a fixed momentum π and the other with an opposite momentum $-\pi$. There has been a construction of the multi-spin giant magnon solution with infinite J_1 and finite $J_2 = J_3$ [26]. From the Polyakov action for strings on $R \times S^5$ in the conformal gauge the three-spin giant magnon solution in the SU(3) sector has been constructed [30] by generalizing the Neumann-Rosochatius ansatz [31], where it is interpreted as a superposition of the bound state of J_2 magnons with a total momentum p_2 and the bound state of J_3 magnons with a different total momentum p_3. The S-matrix for bound states with an arbitrary number of magnons in the SU(2) sector has been investigated in both string and gauge theory sides [32].

In the SL(2) sector using the Nambu-Goto action for strings in $AdS_3 \times S^1$ [24] and that with NS-NS B field [33], the two-spin giant magnon solutions have been presented in the static gauge. In order to take an attempt to extend the SL(2) sector to the larger sector we will use the Polyakov action for strings in $AdS_3 \times S^3$ in the conformal gauge to construct a three-spin giant magnon with one angular momentum in AdS_3 and two angular momenta in S^3. We will see that the two Virasoro constraints connect the two subsectors SL(2) and SU(2) and make an important role to determine the form of the rotating open string configuration. We will discuss a relation between a giant magnon solution on AdS_2 and the sinh-Gordon soliton.

2. Three-spin giant magnons

We consider a three-spin giant magnon in $AdS_5 \times S^5$ which has one spin S in AdS_5 and two spins J_1 and J_2 in S^5. The relevant metric is that of $AdS_3 \times S^3$ part of $AdS_5 \times S^5$

$$ds^2 = -\cosh^2 \rho dt^2 + d\rho^2 + \sinh^2 \rho d\varphi^2 + d\theta^2 + \cos^2 \theta d\phi_1^2 + \sin^2 \theta d\phi_2^2, \tag{2.1}$$

so that the Polyakov string action in the conformal gauge becomes

$$I = -\frac{\sqrt{\lambda}}{4\pi} \int d\tau d\sigma [-\cosh^2 \rho (t'^2 - \dot{t}^2) + \rho'^2 - \dot{\rho}^2 + \sinh^2 \rho (\varphi'^2 - \dot{\varphi}^2)$$

$$+\theta'^2 - \dot\theta^2 + \cos^2\theta(\phi'^2_1 - \dot\phi_1^2) + \sin^2\theta(\phi'^2_2 - \dot\phi_2^2)], \tag{2.2}$$

where the dot and prime denote the derivatives with respect to τ and σ which ranges from $-\infty$ to ∞. We make the ansatz for a rotating open string soliton

$$\begin{aligned} t &= \tau + h_1(y), & \rho &= \rho(y), & \varphi &= \omega(\tau + h_2(y)), \\ \phi_1 &= \tau + g_1(y), & \theta &= \theta(y), & \phi_2 &= w(\tau + g_2(y)), \end{aligned} \tag{2.3}$$

where $y = \sigma - v\tau$. The equations of motion for ϕ_1, ϕ_2 lead to

$$\partial_y g_1 = \frac{v}{1-v^2}\tan^2\theta, \qquad \partial_y g_2 = -\frac{v}{1-v^2}, \tag{2.4}$$

which give the equation of motion for θ

$$(1-v^2)^2\partial_y^2\theta = \sin\theta\cos\theta\left(1 - w^2 - \frac{v^2}{\cos^4\theta}\right). \tag{2.5}$$

The first integral of (2.5) with an appropriate integration constant is obtained by

$$\begin{aligned} (1-v^2)^2(\partial_y\theta)^2 &= \sin^2\theta\left(1 - w^2 - \frac{v^2}{\cos^2\theta}\right) \\ &= (1-w^2)\tan^2\theta(\alpha^2 - \sin^2\theta) \end{aligned} \tag{2.6}$$

with $\alpha = \sqrt{(1-v^2-w^2)/(1-w^2)}$, whose solution is given by

$$\sin\theta = \frac{\alpha}{\cosh\beta y}, \qquad \text{for } -\infty < y < \infty, \tag{2.7}$$

where $\beta = \sqrt{1-v^2-w^2}/(1-v^2), 1-v^2-w^2 \geq 0, 1-w^2 \geq 0$. The angle ϕ_2 is also expressed as $\phi_2 = w(\tau - v\sigma)/(1-v^2)$. These expressions were presented in [23] by using the conformal gauge supplemented by the static choice $t = \tau$ for the Polyakov action of strings in $R \times S^3$.

The equations of motion for t and φ are given by

$$\begin{aligned} \partial_y[(v + (1-v^2)\partial_y h_1)\cosh^2\rho] = 0, \\ \partial_y[(v + (1-v^2)\partial_y h_2)\sinh^2\rho] = 0, \end{aligned} \tag{2.8}$$

which generate two conservation laws

$$\partial_y h_1 = \frac{1}{1-v^2}\left(-v + \frac{c_1}{\cosh^2\rho}\right), \qquad \partial_y h_2 = \frac{1}{1-v^2}\left(-v + \frac{c_2}{\sinh^2\rho}\right), \tag{2.9}$$

where c_1 and c_2 are integration constants. We use the expressions in (2.9) to obtain the equation of motion for ρ

$$(1-v^2)^2\partial_y^2\rho = -\sinh\rho\cosh\rho\left[\omega^2\left(1 - \frac{c_2^2}{\sinh^4\rho}\right) - \left(1 - \frac{c_1^2}{\cosh^4\rho}\right)\right], \tag{2.10}$$

which is compared with (2.5).

We first express one constraint on the energy-momentum tensor of the system $T_{\tau\sigma}$ in terms of $z = \sin\theta$ as

$$\frac{1}{v}(1 - v\partial_y h_1)\partial_y h_1 \cosh^2\rho + (\partial_y\rho)^2 - \frac{\omega^2}{v}(1 - v\partial_y h_2)\partial_y h_2 \sinh^2\rho$$
$$+(\partial_y\theta)^2 - \frac{1}{1-v^2}\left(1 - \frac{v^2}{1-v^2}\frac{z^2}{1-z^2}\right)z^2 + \frac{w^2 z^2}{(1-v^2)^2} = 0, \tag{2.11}$$

which combines with (2.6) to be

$$\frac{1}{v}(1 - v\partial_y h_1)\partial_y h_1 \cosh^2\rho + (\partial_y\rho)^2 - \frac{\omega^2}{v}(1 - v\partial_y h_2)\partial_y h_2 \sinh^2\rho = 0. \tag{2.12}$$

The other constraint $T_{\tau\tau} + T_{\sigma\sigma} = 0$ now reads

$$-[(1 - v\partial_y h_1)^2 + (\partial_y h_1)^2]\frac{\cosh^2\rho}{1+v^2} + (\partial_y\rho)^2 + \omega^2[(1 - v\partial_y h_2)^2 + (\partial_y h_2)^2]\frac{\sinh^2\rho}{1+v^2} \tag{2.13}$$
$$+(\partial_y\theta)^2 + (1-z^2)\left[\frac{1}{1+v^2}\left(1 - \frac{2v^2}{1-v^2}\frac{z^2}{1-z^2}\right) + \frac{z^4 v^2}{(1-v^2)^2(1-z^2)^2}\right] + \frac{w^2 z^2}{(1-v^2)^2} = 0,$$

which also turns out to be of the form

$$-[(1 - v\partial_y h_1)^2 + (\partial_y h_1)^2]\frac{\cosh^2\rho}{1+v^2} + (\partial_y\rho)^2 + \omega^2[(1 - v\partial_y h_2)^2 + (\partial_y h_2)^2]\frac{\sinh^2\rho}{1+v^2} + \frac{1}{1+v^2} = 0 \tag{2.14}$$

owing to (2.6). We require that the equation (2.12) should be identically equal to (2.14). Eliminating the $(\partial_y\rho)^2$ term in (2.12) and (2.14) by subtraction we have one relation expressed in terms of $\partial_y h_1$ and $\partial_y h_2$. The substitution of (2.9) into this extracted relation yields

$$c_1 - \omega^2 c_2 = v. \tag{2.15}$$

In what follows we will take a simple case $c_1 = v, c_2 = 0$. From (2.3) and (2.9) with $c_1 = v$ we have

$$\frac{dt}{d\tau} = \frac{\cosh^2\rho - v^2}{(1-v^2)\cosh^2\rho} > 0, \tag{2.16}$$

which insures forward propagation in time. Thus the equations (2.12) and (2.14) become the same expression

$$\begin{aligned}(1-v^2)^2(\partial_y\rho)^2 &= \sinh^2\rho\left(1 - \omega^2 - \frac{v^2}{\cosh^2\rho}\right)\\ &= (1-\omega^2)\tanh^2\rho\left(\sinh^2\rho + \frac{1-v^2-\omega^2}{1-\omega^2}\right),\end{aligned} \tag{2.17}$$

which is indeed the first integral of the equation of motion for ρ (2.10) and depends on the two parameters v and ω in the form similar to (2.6). Since the $1 - \omega^2 \leq 0$ region is not allowed in view of the expression (2.17), there are two parameter regions

$$\begin{aligned}\mathrm{A}: &\quad 1 - v^2 - \omega^2 \leq 0,\ 1 - \omega^2 \geq 0,\\ \mathrm{B}: &\quad 1 - v^2 - \omega^2 \geq 0,\ 1 - \omega^2 \geq 0,\end{aligned} \tag{2.18}$$

which are also expressed as A: $0 \le 1-\omega^2 \le v^2$ and B: $v^2 \le 1-\omega^2$.

For the region A the eq. (2.17) leads to

$$\partial_y \rho = \pm \frac{\sqrt{1-\omega^2}}{1-v^2} \tanh\rho \sqrt{\sinh^2\rho - \tilde{\alpha}^2}, \qquad \tilde{\alpha} = \sqrt{\frac{v^2+\omega^2-1}{1-\omega^2}}, \tag{2.19}$$

which can be integrated as

$$\frac{1}{\tilde{\alpha}}\sqrt{\sinh^2\rho - \tilde{\alpha}^2} = \begin{cases} \tan\tilde{\beta}y, & \text{for } 0 \le y \le \frac{\pi}{2\tilde{\beta}}, \\ -\tan\tilde{\beta}y, & \text{for -}\frac{\pi}{2\tilde{\beta}} \le y \le 0 \end{cases} \tag{2.20}$$

with $\tilde{\beta} = \sqrt{v^2+\omega^2-1}/(1-v^2)$. This solution lies within a finite range of y and is expressed as

$$\sinh\rho = \frac{\tilde{\alpha}}{\cos\tilde{\beta}y}, \qquad \text{for } -\frac{\pi}{2\tilde{\beta}} \le y \le \frac{\pi}{2\tilde{\beta}}. \tag{2.21}$$

At the boundaries of range $y = \pm\frac{\pi}{2\tilde{\beta}}$ the radial coordinate ρ extends to the infinity, and at $y=0$ it becomes the shortest value $\rho_0 = \sinh^{-1}\tilde{\alpha}$, which is compared with the maximum value $\theta_{\max} = \sin^{-1}\alpha$ at $y=0$ for solution (2.7). In the region A we cannot make a $\omega=0$ reduction that corresponds to the string solution in $AdS_2 \times S^3$, while the $\omega = 0$ reduction is possible in the region B.

For the region B through an appropriate integration constant the eq. (2.17) is similarly integrated as

$$\sinh\rho = \begin{cases} \frac{\hat{\alpha}}{\sinh\hat{\beta}y}, & \text{for } 0 \le y < \infty, \\ -\frac{\hat{\alpha}}{\sinh\hat{\beta}y}, & \text{for } -\infty < y \le 0, \end{cases} \tag{2.22}$$

where

$$\hat{\alpha} = \sqrt{\frac{1-v^2-\omega^2}{1-\omega^2}}, \qquad \hat{\beta} = \frac{\sqrt{1-v^2-\omega^2}}{1-v^2}. \tag{2.23}$$

At $y=0$ ρ extends infinitely to the boundary of AdS_3, while at $y=\pm\infty$ ρ becomes zero to reach the origin of AdS_3. Since this solution is supported in the same infinite range $-\infty < y < \infty$ as that for the solution (2.7), we will analyze the string solution in the parameter region B, which describes an open string on a plane.

The rotating open string is characterized by the following energy and spins

$$\begin{aligned} E &= \frac{\sqrt{\lambda}}{2\pi}\int d\sigma \cosh^2\rho \left(1 + \frac{v^2}{1-v^2}\tanh^2\rho\right), \\ J_1 &= \frac{\sqrt{\lambda}}{2\pi}\int d\sigma \cos^2\theta \left(1 - \frac{v^2}{1-v^2}\tan^2\theta\right), \\ J_2 &= \frac{\sqrt{\lambda}}{2\pi}\frac{w}{1-v^2}\int d\sigma \sin^2\theta, \\ S &= \frac{\sqrt{\lambda}}{2\pi}\frac{\omega}{1-v^2}\int d\sigma \sinh^2\rho, \end{aligned} \tag{2.24}$$

where J_1, J_2 and S are the spins associated with the ϕ_1, ϕ_2 and φ directions. They combine to yield a relation

$$E - J_1 = \frac{S}{\omega} + \frac{J_2}{w}. \tag{2.25}$$

When the solutions (2.22) and (2.7) are plugged into the respective expressions in (2.24) we see that both E and J_1 diverge owing to the effectively long open string configuration, while the difference $E - J_1$ has no such IR divergence but contains the UV divergence. The solution (2.7) is used to rewrite the J_2/w term in (2.25) as a finite value [23, 24]

$$\frac{J_2}{w} = \sqrt{J_2^2 + \frac{\lambda}{\pi^2}\alpha^2}, \tag{2.26}$$

where α is characterized by an angle difference between the two endpoints of the open string as

$$\Delta\phi_1 = \int_{-\infty}^{\infty} dy \partial_y g_1(y) = 2\cos^{-1}\sqrt{1-\alpha^2}, \tag{2.27}$$

which is identified with the magnon momentum and yields $\alpha = \sin\frac{\Delta\phi_1}{2}$.

On the other hand the S spin contribution to the string energy is expressed through (2.17) and the change of variable $z = \cosh\rho$ as

$$\frac{S}{\omega} = \frac{\sqrt{\lambda}}{2\pi}\frac{2}{1-v^2}\int_{\infty}^{0} d\rho \frac{\sinh^2\rho}{\partial_y\rho} = \frac{\sqrt{\lambda}}{\pi}\frac{1}{\sqrt{1-\omega^2}}\int_1^{\infty} dz \frac{z}{\sqrt{z^2 - z_0^2}}, \tag{2.28}$$

where $z_0 = v/\sqrt{1-\omega^2}$ and $\partial_y\rho = -\sqrt{1-\omega^2}\tanh\rho\sqrt{\sinh^2\rho + \hat{\alpha}^2}/(1-v^2)$ for $0 \le y < \infty$. However, this integration diverges because the rotating long string stretches to the boundary of AdS_3. By introducing a cutoff Λ to regulate the UV divergence we evaluate (2.28) as

$$\frac{S}{\omega} = \frac{\sqrt{\lambda}}{\pi}\frac{1}{\sqrt{1-\omega^2}}(\Lambda - \sqrt{1-z_0^2}). \tag{2.29}$$

This divergence appeared when the giant magnon solution was derived by analyzing the Nambu-Goto action in the static gauge for the string with two spins J_1 and S in $AdS_3 \times S^1$ [24]. Following the prescription of ref. [24] we subtract the divergent term to have a regulated value

$$\frac{S_{\text{reg}}}{\omega} = -\frac{\sqrt{\lambda}}{\pi}\sqrt{\frac{1-z_0^2}{1-\omega^2}}, \tag{2.30}$$

from which ω is obtained by

$$\omega = \frac{|S_{\text{reg}}|}{\sqrt{S_{\text{reg}}^2 + \frac{\lambda}{\pi^2}(1-z_0^2)}}. \tag{2.31}$$

Combining (2.30) and (2.31) the S spin contribution is expressed as a magnon-like dispersion relation

$$\frac{S_{\text{reg}}}{\omega} = -\sqrt{S_{\text{reg}}^2 + \frac{\lambda}{\pi^2}\hat{\alpha}^2}, \tag{2.32}$$

which resembles (2.26), where $\hat{\alpha}$ and α are similarly defined by using ω and w respectively. Here the parameter $\hat{\alpha}$ is characterized by a time difference between the two endpoints of the open string

$$\Delta t = \int_{-\infty}^{\infty} dy \partial_y h_1(y) = -2\tan^{-1}\frac{\sqrt{1-\hat{\alpha}^2}}{\hat{\alpha}}, \tag{2.33}$$

which reduces to $\hat{\alpha} = \cos\frac{\Delta t}{2}$. Thus we have a dispersion relation for the three-spin giant magnon in $AdS_3 \times S^3$

$$(E - J_1)_{\rm reg} = -\sqrt{S_{\rm reg}^2 + \frac{\lambda}{\pi^2}\cos^2\frac{\Delta t}{2}} + \sqrt{J_2^2 + \frac{\lambda}{\pi^2}\sin^2\frac{\Delta\phi_1}{2}}, \tag{2.34}$$

which is regarded as the energy of a superposition of a bound state of J_2 magnons with momentum $\Delta\phi_1$ and a bound state of $|S_{\rm reg}|$ magnons with momentum $\pi + \Delta t$. Up to a negative sign this dispersion relation has the similar structure to that for the giant magnon with the three spins J_1, J_2, J_3 in $R \times S^5$ [29, 30].

3. Giant magnons on AdS_2 and sinh-Gordon solitons

Let us analyze the AdS_3 part of the three-spin giant magnon configuration. The AdS_3 space-time is parametrized by a complex two-component vector $Y_i = (Y_0, Y_1)$

$$Y_0 = \cosh\rho e^{it}, \qquad Y_1 = \sinh\rho e^{i\varphi}, \tag{3.1}$$

which obeys $Y_i^* Y^i = -1$, $Y^i = \eta^{ij} Y_j$, with $\eta^{ij} = \mathrm{diag}(-1, 1)$. Alternatively, a real four-component vector

$$n_i = (\cosh\rho\cos t, \cosh\rho\sin t, \sinh\rho\cos\varphi, \sinh\rho\sin\varphi) \tag{3.2}$$

parametrizes the AdS_3 space-time in such a way that $n_i n^i = -1$ and $n^i = \eta^{ij} n_j$ with the flat $R^{2,2}$ metric $\eta^{ij} = \mathrm{diag}(-1, -1, 1, 1)$. The reduced Virasoro constraint (2.14) for the string motion in AdS_3 can be expressed in a compact form as

$$\dot{n}_i \dot{n}^i + n'_i n'^i = -1. \tag{3.3}$$

Here in order to capture a fascinating feature of the giant magnon on AdS_3 we try to compute a combination

$$\dot{n}_i \dot{n}^i - n'_i n'^i = -\cosh^2\rho(\dot{t}^2 - t'^2) + \dot{\rho}^2 - \rho'^2 + \sinh^2\rho(\dot{\varphi}^2 - \varphi'^2). \tag{3.4}$$

By substituting the solution (2.22) into the first equation in (2.9) and integrating it we express the time coordinate t as

$$t - \tau = h_1 = -\tan^{-1}\left(\frac{\sqrt{1-\hat{\alpha}^2}}{\hat{\alpha}}\tanh\hat{\beta}y\right) + k, \tag{3.5}$$

which reproduces (2.33). If the integration constant k is chosen to be zero, it is convenient to express (3.5) as

$$\cot(t - \tau) = -\frac{\hat{\alpha}}{\sqrt{1-\hat{\alpha}^2}}\coth\hat{\beta}y. \tag{3.6}$$

The eq. (3.6) combines with (2.22) to yield

$$\cosh\rho = \begin{cases} \frac{\sqrt{1-\hat{\alpha}^2}}{\sin(t-\tau)}, & \text{for } 0 \le t-\tau \le \hat{h}_1 < \frac{\pi}{2}\ (-\infty < y \le 0), \\ -\frac{\sqrt{1-\hat{\alpha}^2}}{\sin(t-\tau)}, & \text{for -}\frac{\pi}{2} < -\hat{h}_1 \le t-\tau \le 0\ (0 \le y < \infty), \end{cases} \tag{3.7}$$

where $\hat{h}_1 = \tan^{-1}(\sqrt{1-\hat{\alpha}^2}/\hat{\alpha})$. By substituting the derivatives of (2.22) and (3.6) with respect to τ and σ into (3.4) and taking account of (3.7) and $\partial_y h_2 = -v/(1-v^2)$ we find a concise expression

$$\dot{n}_i \dot{n}^i - n'_i n'^i = -\left(1 + \frac{2(1-\omega^2)\hat{\alpha}^2}{1-v^2} \frac{1}{\sinh^2 \hat{\beta} y}\right). \tag{3.8}$$

It can be checked that the constraint (3.3) is indeed satisfied by the explicit relations (3.6) and (3.7). Alternatively, instead of the explicit relations we directly use (2.9) to express (3.4) as

$$\begin{aligned} \dot{n}_i \dot{n}^i - n'_i n'^i &= -\cosh^2 \rho - \frac{2v^2}{1-v^2} \sinh^2 \rho \\ &+ (1-v^2) \left[\frac{v^2}{(1-v^2)^2} \frac{\sinh^4 \rho}{\cosh^2 \rho} - (\partial_y \rho)^2 + \frac{\omega^2}{(1-v^2)^2} \sinh^2 \rho \right], \end{aligned} \tag{3.9}$$

which reduces to (3.8) through the substitution of (2.22).

If we define a scalar field ϕ as

$$\cosh 2\phi = -(\dot{n}_i \dot{n}^i - n'_i n'^i), \tag{3.10}$$

whose ϕ is real bcause of comparing $\cosh 2\phi > 1$ with (3.8), we obtain

$$\sinh \phi = \sqrt{\frac{1-v^2-\omega^2}{1-v^2}} \frac{1}{\sinh \hat{\beta} y}. \tag{3.11}$$

From the expression (3.11) the scalar field ϕ is shown to obey the following equation

$$\partial_y^2 \phi = \frac{1}{1-v^2} \frac{\sinh \phi}{\cosh^3 \phi} \left(\sinh^4 \phi + 2 \sinh^2 \phi + \frac{1-v^2-\omega^2}{1-v^2} \right). \tag{3.12}$$

When considering a special $\omega = 0$ case, that is, the giant magnon on AdS_2, we have

$$\sinh \phi = \frac{1}{\sinh \frac{y}{\sqrt{1-v^2}}} \tag{3.13}$$

and the eq. (3.12) implies that the scalar field ϕ satisfies the sinh-Gordon equation

$$(\partial_\tau^2 - \partial_\sigma^2) 2\phi = -\sinh 2\phi. \tag{3.14}$$

Thus we see that there is a relation between the giant magnon on AdS_2 and the sinh-Gordon soliton, which corresponds to the map between the giant magnon on S^2 and the sine-Gordon soliton [14].

Moreover, it is instructive to substitute the explicit solutions (2.22), (3.6) and (3.7) with $\varphi = \omega(\tau - v\sigma)/(1-v^2)$ into the complex coordinates Y_0 and Y_1 in (3.1) and then obtain

$$\begin{aligned} Y_0 &= \pm e^{i\tau} \left[i\sqrt{1-\hat{\alpha}^2} - \hat{\alpha} \coth\left(\hat{\beta}_0 \frac{y}{\sqrt{1-v^2}} \right) \right], \\ Y_1 &= \mp \frac{\hat{\alpha}}{\sinh\left(\hat{\beta}_0 \frac{y}{\sqrt{1-v^2}} \right)} e^{i\sqrt{1-\hat{\beta}_0^2}(\tau - v\sigma)/\sqrt{1-v^2}}, \qquad \hat{\beta}_0 = \sqrt{\frac{1-v^2-\omega^2}{1-v^2}}, \end{aligned} \tag{3.15}$$

whose signs correspond to $-\infty < y \leq 0$ and $0 \leq y < \infty$ respectively. This expression for the string solution in AdS_3 looks similar to that expressed by the two complex coordinates for the string solution in $R \times S^3$ corresponding to the dyonic giant magnon with two independent angular momenta which was shown to be related with the charged soliton of the complex sine-Gordon equation [22].

4. Conclusion

We have used the conformal gauge for the Polyakov action of strings in $AdS_3 \times S^3$ to construct the three-spin giant magnon solution with one spin in AdS_3 and two spins in S^3. By taking advantage of the explicit expression for the giant magnon solution we have demonstrated a mapping between the giant magnon on AdS_2 and the sinh-Gordon soliton.

We have observed that the string configuration in S^3 makes an effect on the string motion in AdS_3 indirectly through the two Virasoro constraints. From the Polyakov action of strings in $R \times S^3$ in the conformal gauge the static choice $t = \tau$ was used [23] to construct the two-spin giant magnon solution in the SU(2) sector, while for the three-spin giant magnon in $AdS_3 \times S^3$ this choice has not been allowed such that the string time coordinate has a nontrivial dependence on the worldsheet coordinates τ and σ. The arbitrary parameters c_1 and c_2 that characterize the time and angle coordinates of string in AdS_3 have been chosen so as to satisfy the two Virasoro constraints. By regularizing the UV divergence arising from the configuration of the string stretched to the boundary of AdS_3, the dispersion relation of the three-spin giant magnon has been obtained as the energy of a superposition of two bound states of magnons. In the SU(2) subsector one bound state with J_2 magnons has the total momentum which is given by the difference of the angle coordinate that is associated with infinite spin J_1, whereas in the SL(2) subsector the other bound state with $|S_{\rm reg}|$ magnons has the total momentum which is specified by the difference of the time coordinate that is associated with infinite energy E.

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